Application of the Decomposition Method of Adomian for Solving the Pantograph Equation of Order *m*

Fatemeh Shakeri and Mehdi Dehghan

Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Avenue, Tehran 15914, Iran

Reprint requests to M. D.; E-mail: mdehghan@aut.ac.ir

Z. Naturforsch. **65a,** 453 – 460 (2010); received June 30, 2009 / revised September 16, 2009

In many fields of the contemporary science and technology, systems with delaying links often appear. By a delay differential equation (DDE), we mean an evolutionary system in which the (current) rate of change of the state depends on the historical status of the system. Delay models play a relevant role in different fields such as biology, economy, control, and electrodynamics and hence have been attracted a lot of attention of the researchers in recent years. In this study, the numerical solution of a well-known delay differential equation, namely, the pantograph equation is investigated by means of the Adomian decomposition method and then a numerical evaluation is included to demonstrate the validity and applicability of this procedure.

Key words: Delay Differential Equations; Pantograph Equation of Order *m*; Adomian Decomposition Method; Semi-Analytical Approach.

1. Introduction

Delay differential equations (DDEs) provide a powerful means of modelling many phenomena in applied sciences such as medicine, economy, biology, and electrodynamics. They allow in fact a mean of modelling phenomena, where the rate of variation of a quantity does not only depend on the value of the quantity itself at time t but also on previous values, that is on its history.

In the last decades, the computational methods for DDEs have been studied by many authors, and a significant number of important results have been found. However, as an important case of DDEs, the study of the pantograph equation has been developed rapidly and growing attention has being paid to its analysis and numerical solution.

A pantograph is a device that collects electric current from overhead lines for electric trains or trams. The term pantograph derives from the resemblance to pantograph devices for copying writing and drawings.

The pantograph equation originated from the work of Ockendon and Taylor [1] which models and redesigns overhead electricity collection system for a train to ensure the contact to the wires. For more comprehensive discussion of this equation we refer the reader to the papers [2 – 13].

In the present work, we consider the following nonhomogeneous pantograph equation of order m:

$$u^{(m)}(t) = \sum_{j=0}^{J} \sum_{k=0}^{m-1} \mu_{jk}(t) u^{(k)}(q_{jk}t) + f(t),\ t \geq 0, \quad (1)$$

with the initial conditions

$$u^{(k)}(0) = \gamma_k, \quad k = 0, 1, \ldots, m-1,$$

where $0 < q_{jk} \leq 1$, μ_{jk} are known functions, and $.^{(m)}$ in $u^{(m)}$ is considered as mth derivation of the function u with respect to the variable t.

The solution of this equation is constructed by means of the Adomian decomposition method. In recent years a large amount of literature developed concerning the Adomian decomposition method and the related modification. This method was first proposed by the American mathematician G. Adomian (1923 – 1996) and has been applied already to a wide class of stochastic and deterministic problems in science and engineering. It is based on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using Adomian polynomials [14, 15]. Wazwaz [16] developed a framework to obtain exact solutions to Fisher's equation and to a nonlinear diffusion equation of Fisher type by employing the Adomian decomposition method. Author of [17] presented a reliable algorithm for solving linear and nonlinear Schrödinger equations which is based on Adomian's decomposition method. The Adomian decomposition method is employed in [18] to find exact and approximate solutions for a nonlinear dispersive Korteweg-de Vries (KdV) equation with an initial profile. Single-soliton, two-soliton, and rational solutions are obtained by this approach. Numerical examples are given to illustrate the pertinent feature of the algorithm proposed. Authors of [19] used the Adomian decomposition method and the phenomenon of the self-canceling noise terms for solving weakly singular Volterra-type integral equations. The solution is given by a convergent power series. A comparison of the given method with collocation-type methods that use a nonpolynomial basis shows the effectiveness of the employed method. In [20] the Adomian decomposition method and the modified Adomian decomposition method are improved for solving approximately homogeneous and inhomogeneous two-dimensional heat equations using Padé approximations. Also this approach was used to find approximate solutions of the coupled Burgers equations. The results show that these techniques increase efficiently the accuracy of approximate solutions and lead to convergence with a rate faster than using the Adomian decomposition and the modified Adomian decomposition methods [20]. Also employing higher-order Padé approximations produces more efficient results. Adomian's decomposition method is proposed in [21] to approximate the solutions of the nonlinear damped generalized regularized long-wave equation with a variable coefficient.

This method is applied actively on various differential equations such as the reaction convection diffusion equation [22], Laplace equation [23], generalized nonlinear Boussinesq equation [24], Fisher's equation [16], Burgers' equation with fractional derivation [25], Camassa-Holm equation [26], Navier-Stokes equations [27], Emden-Fowler type of equations and wave-type equation with singular behaviour [28], Kawahara equation [29], some problems in calculus of variations [30], system of fractional differential equations [31], Fokker-Planck equation [32], hyperbolic partial differential equations [33], and many other problems in science and engineering. This approach is useful for obtaining both a closed form and the explicit solution and numerical approximations of linear or nonlinear differential equations and it is also quite straightforward to write computer codes. In this work, we employ this technique for solving the pantograph equation (1).

The structure of this paper is as follows:

In Section 2, a brief description of the Adomian decomposition method and its application to the pantograph equation (1) are provided. Section 3 contains some numerical examples to show efficiently and applicability of the new technique for the studied problem. Finally a conclusion is drawn in Section 4.

2. Adomian Decomposition Method

In this section we apply the decomposition procedure of Adomian to solve equation (1).

Consider the pantograph equation (1). If we define the operator $L = \frac{\partial^m}{\partial t^m}$, then equation (1) can be written as

$$L(u(t)) = \sum_{j=0}^{J} \sum_{k=0}^{m-1} \mu_{jk}(t) u^{(k)}(q_{jk}t) + f(t). \tag{2}$$

Assume that the inverse operator L^{-1} exists and can be taken as follows:

$$L^{-1}(.) = \int_0^t \int_0^{s_{m-1}} \cdots \int_0^{s_1} (.) \mathrm{d}s \mathrm{d}s_1 \ldots \mathrm{d}s_{m-1}.$$

Applying the inverse operator L^{-1} to both sides of (2) yields

$$L^{-1}Lu(t) = L^{-1}\left(\sum_{j=0}^{J} \sum_{k=0}^{m-1} \mu_{jk}(t) u^{(k)}(q_{jk}t) \right) + L^{-1}(f(t)),$$

and therefore we can write

$$\begin{aligned} u(t) &= \sum_{l=0}^{m-1} \gamma_l \frac{t^l}{l!} \\ &+ L^{-1}\left(\sum_{j=0}^{J} \sum_{k=0}^{m-1} \mu_{jk}(t) u^{(k)}(q_{jk}t) \right) + L^{-1}(f(t)). \end{aligned} \tag{3}$$

Based on the Adomian decomposition method, we seek the solution of (1) in the following form:

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{4}$$

where the components u_n satisfy the following recursive relationships:

$$u_0(t) = \sum_{l=0}^{m-1} \gamma_i \frac{t^l}{l!} + L^{-1}(f(t)), \tag{5}$$

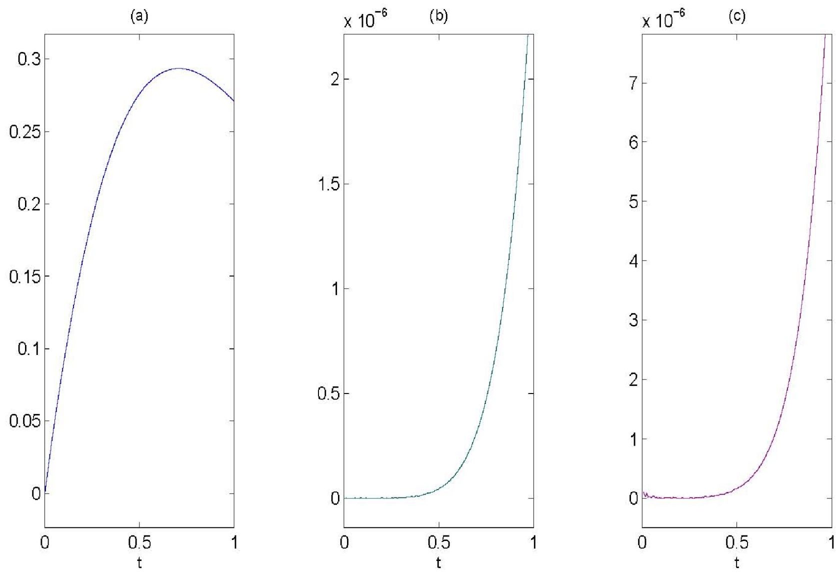

Fig. 1. Plot of (a) the exact solution, (b) the absolute error $|u(t)-\phi_3(t)|$, and (c) the relative error $\left|\frac{u(t)-\phi_3(t)}{u(t)}\right|$ of Example 1.

$$u_n(t) = L^{-1}\left(\sum_{j=0}^{J}\sum_{k=0}^{m-1}\mu_{jk}(t)u_{n-1}^{(k)}(q_{jk}t)\right), \quad n \geq 1. \tag{6}$$

The n-term approximation of the solution is defined as $\phi_n(t) = \sum_{k=0}^{n} u_k(t)$ and $u = \lim_{n\to\infty}\phi_n$. As we know, the more terms we add to the approximate solution, the more accurate it will be. Convergence of Adomian's decomposition scheme was established by many authors using the fixed point theorem, see for example [34].

3. Test Problems

In this section, we present some examples with analytical solution to support the performance of the method for solving equation (1).

3.1. Example 1

In this example, consider equation (1) with $m = 1$, $J = 3$, and the following nonzero coefficients μ_{jk} and q_{jk}:

$$\mu_{00}(t) = 3\exp\left(\frac{1}{3}t\right), \quad \mu_{10} = 2\exp\left(\frac{2}{3}t\right),$$
$$\mu_{20} = 1, \quad \mu_{30} = -3\exp\left(-\frac{3}{4}t\right),$$
$$q_{00} = \frac{1}{6}, \quad q_{10} = \frac{1}{3}, \quad q_{20} = \frac{1}{5}, \quad q_{30} = \frac{1}{4},$$

and

$$f(t) = \exp(-2t)(-2t^2+1) - \left(\frac{1}{25}t^2 + \frac{1}{5}t\right)\exp\left(-\frac{2}{5}t\right) + \left(\frac{3}{16}t^2 + \frac{3}{4}t\right)\exp\left(-\frac{5}{4}t\right) - \frac{11}{36}t^2 - \frac{7}{6},$$

with the initial condition

$$u(0) = 0.$$

The exact solution of this equation is

$$u(t) = (t^2+t)\exp(-2t).$$

Using the discussion presented in Section 2, we obtain the recurrent relations

$$u_0(t) = \int_0^t \Bigg[\exp(-2s)(-2s^2+1) + \left(-\frac{1}{25}s^2 - \frac{1}{5}s\right)\exp\left(-\frac{2}{5}s\right) + \left(\frac{3}{16}s^2 + \frac{3}{4}s\right)\exp\left(-\frac{5}{4}s\right) - \frac{11}{36}s^2 - \frac{7}{6}\Bigg]\mathrm{d}s, \tag{7}$$

$$u_n(t) = \int_0^t \Bigg[3\exp\left(\frac{1}{3}s\right)u_{n-1}\left(\frac{1}{6}s\right) + 2\exp\left(\frac{2}{3}s\right)u_{n-1}\left(\frac{1}{3}s\right) + u_{n-1}\left(\frac{1}{5}s\right) - 3\exp\left(-\frac{3}{4}s\right)u_{n-1}\left(\frac{1}{4}s\right)\Bigg]\mathrm{d}s, \quad n \geq 1. \tag{8}$$

t	Exact value	Absolute error $\lvert u(t)-\phi_3(t)\rvert$	Relative error $\left\lvert \frac{u(t)-\phi_3(t)}{u(t)} \right\rvert$
0.10	0.09006038283858	1.330515519435808e-9	1.477359386558006e-8
0.20	0.16087681104855	1.145843863805559e-9	7.122492398607637e-9
0.30	0.21403653807667	2.394064863154180e-9	1.118530922181426e-8
0.40	0.25162421990564	1.460423325314420e-8	5.803985506093411e-8
0.50	0.27590958087858	4.429447072717832e-8	1.605398065051999e-7
0.60	0.28914644343571	1.293086575334032e-7	4.472081897218715e-7
0.70	0.29345038709051	3.143727146906390e-7	1.071297665706177e-6
0.80	0.29073098591230	6.910247604223581e-7	2.376852808633199e-6
0.90	0.28266109885891	1.400178832699456e-6	4.953560423956108e-6
1.00	0.27067056647323	2.654101314321301e-6	9.805651751882835e-6

Table 1. Exact value, absolute error $\lvert u(t)-\phi_3(t)\rvert$, and relative error $\left\lvert \frac{u(t)-\phi_3(t)}{u(t)} \right\rvert$ of Example 1.

t	Exact value	Absolute error $\lvert u(t)-\phi_3(t)\rvert$	Relative error $\left\lvert \frac{u(t)-\phi_3(t)}{u(t)} \right\rvert$
0.10	0.08330633010743	1.657182480130359e-8	1.989263574560606e-7
0.20	0.16645071157022	1.451047282330364e-8	8.717579328089601e-8
0.30	0.24927154912043	6.190369106207072e-9	2.483383734746398e-8
0.40	0.33160795341738	1.104694504761028e-7	3.331326928008312e-7
0.50	0.41330009194794	3.640235490820487e-8	8.807729690219947e-8
0.60	0.49418953745640	1.436576582136695e-7	2.906934431535664e-7
0.70	0.57411961308975	3.638991614707265e-7	6.338385820200844e-7
0.80	0.65293573345219	5.016528348278371e-8	7.683035391178710e-8
0.90	0.73048574077257	2.812999468915223e-7	3.850861573204924e-7
1.00	0.80662023540036	3.521479783330506e-7	4.365722094217815e-7

Table 2. Exact value, absolute error $\lvert u(t)-\phi_3(t)\rvert$, and relative error $\left\lvert \frac{u(t)-\phi_3(t)}{u(t)} \right\rvert$ of Example 2.

t	Exact value	Absolute error $\lvert u(t)-\phi_3(t)\rvert$	Relative error $\left\lvert \frac{u(t)-\phi_3(t)}{u(t)} \right\rvert$
0.10	0.90199000000000	1.946773409297845e-9	2.158309304202757e-9
0.20	0.81568000000000	5.218227581694551e-10	6.397395524831491e-10
0.30	0.75157000000000	4.275772993688097e-9	5.689121430722483e-9
0.40	0.71776000000000	2.291937614135427e-9	3.193181027272942e-9
0.50	0.71875000000000	1.843137325843290e-9	2.564364975086317e-9
0.60	0.75424000000000	2.039336426017533e-8	2.703829584770806e-8
0.70	0.81793000000000	8.444837924332957e-8	1.032464627087032e-7
0.80	0.89632000000000	2.939036779111113e-7	3.279003903863701e-7
0.90	0.96751000000000	8.719539396386428e-7	9.012350669643133e-7
1.00	1	2.249344328071934e-6	2.249344328071934e-6

Table 3. Exact value, absolute error $\lvert u(t)-\phi_3(t)\rvert$, and relative error $\left\lvert \frac{u(t)-\phi_3(t)}{u(t)} \right\rvert$ of Example 3.

Having (7) and (8), we can obtain u_n for $n = 0,1,2,3$. In Figure 1 and Table 1, numerical results, the error functions $\lvert u(t)-\phi_3(t)\rvert$ and $\left\lvert \frac{u(t)-\phi_3(t)}{u(t)} \right\rvert$ are presented, respectively. It should be noted that only four terms have been used in evaluating the approximate solutions, but we have achieved a very good approximation with high accuracy.

3.2. Example 2

In this example, assume equation (1) with $m = 2$, $J = 1$, and the following nonzero coefficients μ_{jk} and q_{jk}:

$$\mu_{00}(t) = 1, \quad \mu_{10} = -2\cos\left(-\frac{t}{6}\right),$$
$$\mu_{01} = -4\sin\left(\frac{t}{4}\right),$$
$$q_{00} = 1, \quad q_{10} = \frac{1}{3}, \quad q_{01} = \frac{1}{2},$$

and

$$f(t) = -\frac{1}{4}\sin\left(\frac{t}{2}\right) - \frac{1}{9}\sin\left(\frac{t}{3}\right) + \frac{2}{3}\sin\left(\frac{5}{12}t\right) + \frac{2}{3}\sin\left(\frac{t}{12}\right) + \sin\left(\frac{5}{18}t\right) - \sin\left(\frac{t}{18}\right),$$

with the initial conditions

$$u(0) = 0, \quad u^{(1)}(0) = \frac{5}{6},$$

for which the exact solution is

$$u(t) = \sin\left(\frac{t}{2}\right) + \sin\left(\frac{t}{3}\right).$$

To solve this equation by means of Adomian's approach, with respect to (5) and (6), the components u_n

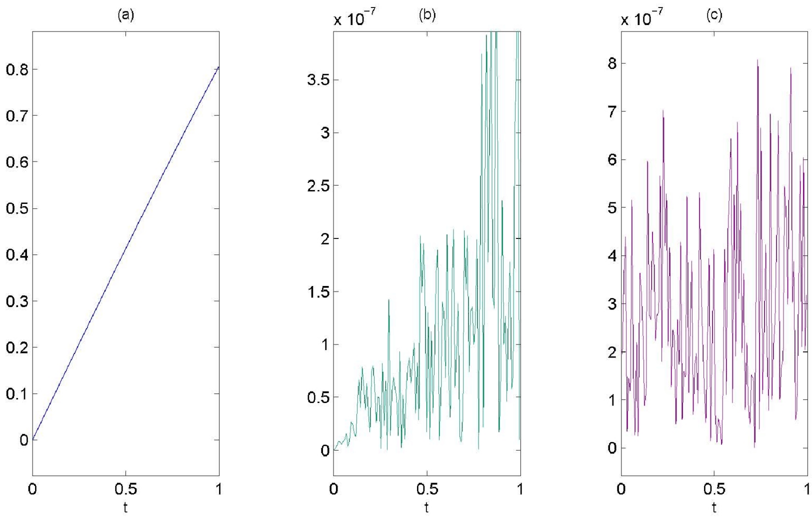

Fig. 2. Plot of (a) the exact solution, (b) the absolute error $|u(t)-\phi_3(t)|$, and (c) the relative error $\left|\frac{u(t)-\phi_3(t)}{u(t)}\right|$ of Example 2.

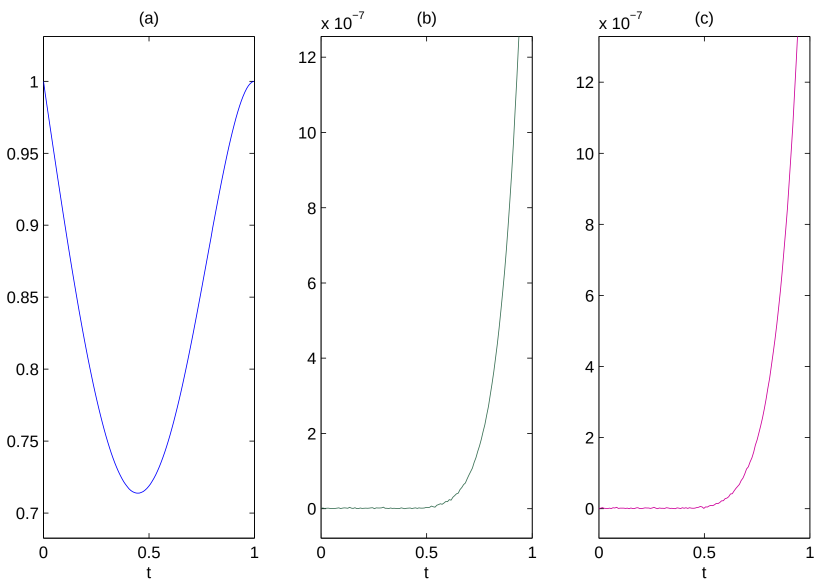

Fig. 3. Plot of (a) the exact solution, (b) the absolute error $|u(t)-\phi_3(t)|$, and (c) the relative error $\left|\frac{u(t)-\phi_3(t)}{u(t)}\right|$ of Example 3.

of the series solution are given as:

$$u_0(t)=\frac{5}{6}t+\int_0^t\int_0^{s_1}\left[-\frac{1}{4}\sin\left(\frac{s}{2}\right)-\frac{1}{9}\sin\left(\frac{s}{3}\right)+\frac{2}{3}\sin\left(\frac{5}{12}s\right)+\frac{2}{3}\sin\left(\frac{s}{12}\right)+\sin\left(\frac{5}{18}s\right)-\sin\left(\frac{s}{18}s\right)\right]\mathrm{d}s\mathrm{d}s_1, \tag{9}$$

$$u_n(t)=\int_0^t\int_0^{s_1}\left(-4\sin\left(\frac{s}{4}\right)u_{n-1}^{(1)}\left(\frac{s}{2}\right)+u_{n-1}-2\cos\left(\frac{s}{6}\right)u_{n-1}\left(\frac{s}{3}\right)\right)\mathrm{d}s\mathrm{d}s_1,\quad n\geq 1. \tag{10}$$

In Figure 2 and Table 2, the results are shown using only four terms of the Adomian decomposition series.

3.3. Example 3

Consider equation (1) with $m=3$, $J=1$ and the following nonzero coefficients μ_{jk} and q_{jk}:

$$\mu_{01}=\exp(t-1),\quad \mu_{11}=-1,\quad \mu_{02}=\frac{t}{3},$$
$$q_{01}=\tfrac{1}{3},\quad q_{11}=\tfrac{1}{2},\quad q_{02}=1,$$

and

t	Exact value	Absolute error $\lvert u(t)-\phi_3(t)\rvert$	Relative error $\left\lvert\frac{u(t)-\phi_3(t)}{u(t)}\right\rvert$
0.10	−1.81858600000000	0.00000149543918	0.00000082230875
0.20	−2.06918400000000	0.00002287242410	0.00001105383770
0.30	−2.24468400000000	0.00011054544822	0.00004924766614
0.40	−2.33833600000000	0.00033308981876	0.00014244737230
0.50	−2.34375000000000	0.00077415217723	0.00033030492895
0.60	−2.75686400000000	0.00105877802041	0.00038405159645
0.70	−3.10671600000000	0.00020206097034	0.00006504005205
0.80	−3.32841600000000	0.00207936965176	0.00062473250091
0.90	−3.33271400000000	0.00380118603394	0.00114056772766
1.00	−3.00000000000000	0.00474593559521	0.00158197853174

Table 4. Exact value, absolute error $|u(t)-\phi_3(t)|$, and relative error $\left|\frac{u(t)-\phi_3(t)}{u(t)}\right|$ of Example 4.

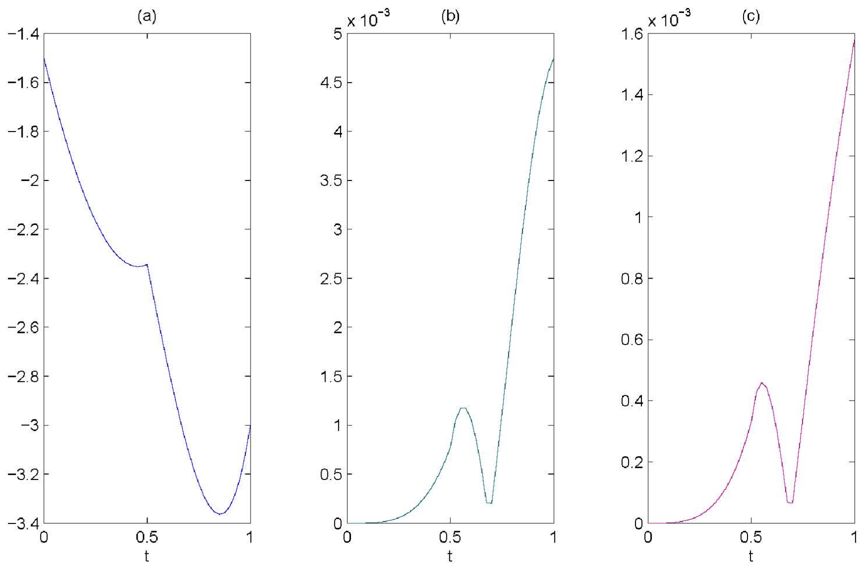

Fig. 4. Plot of (a) the exact solution, (b) the absolute error $|u(t)-\phi_3(t)|$, and (c) the relative error $\left|\frac{u(t)-\phi_3(t)}{u(t)}\right|$ of Example 4.

$$f(t)=\left(-\frac{305}{48}-\frac{5}{81}\exp(t-1)\right)t^4+\left(\frac{125}{2}+\frac{2}{3}\exp(t-1)\right)t^2-\exp(t-1)-11,$$

with the initial conditions

$$u(0)=1,\quad u^{(1)}(0)=-1,\quad u^{(2)}(0)=0.$$

The exact solution of this equation is

$$u(t)=-t^5+2t^3-t+1.$$

By the same manipulation as previous examples, the components u_n are obtained in the following manner:

$$u_0(t)=1-t+\int_0^t\int_0^{s_2}\int_0^{s_1}\left[\left(-\frac{305}{48}-\frac{5}{81}\exp(s-1)\right)s^4+\left(\frac{125}{2}+\frac{2}{3}\exp(s-1)\right)s^2-\exp(s-1)-11\right]\mathrm{d}s\mathrm{d}s_1\mathrm{d}s_2, \quad (11)$$

$$u_n(t)=\int_0^t\int_0^{s_2}\int_0^{s_1}\left[\frac{s}{3}u_{n-1}^{(2)}(s)+\exp(s-1)u_{n-1}^{(1)}\left(\frac{s}{3}\right)-u_{n-1}^{(1)}\left(\frac{s}{2}\right)\right]\mathrm{d}s\mathrm{d}s_1\mathrm{d}s_2,\quad n\geq 1. \quad (12)$$

By these recurrent relations, we calculate u_n for $n=0,1,2,3$. Table 3 and Figure 3 express the numerical results and errors arisen from the current technique in this equation. These results show that the new method produces high accurate approximations only with a few iterations.

3.4. Example 4

Consider equation (1) with $m = 1$, $J = 1$, and the following nonzero coefficients μ_{jk} and q_{jk}:

$$\mu_{00}(t) = -2t, \quad \mu_{10} = -2, \quad q_{00} = \frac{1}{3}, \quad q_{10} = \frac{1}{2},$$

and

$$f(t) = \mathrm{abs}\left(1, t - \frac{1}{2}\right)(t^5 + t^3 - 3t - 1) + \left(\left|t - \frac{1}{2}\right| + 1\right)(5t^4 + 3t^2 - 3) + \left(2\left|\frac{1}{3}t - \frac{1}{2}\right| + 2\right)\left(-t + \frac{1}{27}t^3 \frac{1}{243}t^5 - 1\right)t + \left(3t - \frac{1}{4}t^3 \frac{1}{16}t^5 + 2\right)\left(\left|\frac{1}{2}t - \frac{1}{2}\right| + 1\right),$$

where

$$\mathrm{abs}(1,t) = \begin{cases} 1, t > 0, \\ 0, t = 0, \\ -1, t < 0, \end{cases}$$

with the initial condition

$$u(0) = -\frac{3}{2}.$$

The exact solution of this equation is

$$u(t) = \left(\left|t - \frac{1}{2}\right| + 1\right)(t^5 + t^3 - 3t - 1).$$

According to Adomian's method, through direct calculation, we obtain the following formulas:

$$u_0(t) = -\frac{3}{2} + \int_0^t \mathrm{abs}\left(1, s - \frac{1}{2}\right)(s^5 + s^3 - 3s - 1) + \left(\left|s - \frac{1}{2}\right| + 1\right)(5s^4 + 3s^2 - 3) + \left(2\left|\frac{1}{3}s - \frac{1}{2}\right| + 2\right)\left(-s + \frac{1}{27}s^3 \frac{1}{243}s^5 - 1\right)s + \left(3s - \frac{1}{4}s^3 \frac{1}{16}s^5 + 2\right)\left(\left|\frac{1}{2}s - \frac{1}{2}\right| + 1\right)\mathrm{d}s,$$

$$u_0(t) = -\frac{3}{2} + \begin{cases} \frac{13}{4}t^2 - \frac{89}{96}t^4 - \frac{15859}{15552}t^6 - \frac{1}{2}t + \frac{557}{360}t^5 + \frac{2}{9}t^3 - \frac{1}{2916}t^8 + \frac{113}{181444}t^7, & t \le \frac{1}{2}, \\ -\frac{11}{4}t^2 + \frac{103}{96}t^4 + \frac{15245}{15552}t^6 + \frac{1}{2}t + 1 + \frac{197}{360}t^5 - \frac{7}{9}t^3 - \frac{1}{2916}t^8 + \frac{113}{18144}t^7, & \frac{1}{2} < t \le 1, \\ -\frac{13}{4}t^2 + \frac{109}{96}t^4 + \frac{15407}{15552}t^6 - \frac{3}{2}t + \frac{8353}{3360} + \frac{179}{360}t^5 + \frac{2}{9}t^3 - \frac{1}{2916}t^8 - \frac{7}{2592}t^7, & 1 < t \le \frac{3}{2}, \\ -\frac{9}{4}t^2 + \frac{77}{96}t^4 + \frac{15535}{15552}t^6 - \frac{3}{2}t + \frac{64231}{53760} + \frac{521}{1080}t^5 + \frac{4}{9}t^3 + \frac{1}{2916}t^8 - \frac{211}{54432}t^7, & t > \frac{3}{2}, \end{cases}$$

$$u_n(t) = \int_0^t \left(-2su_{n-1}\left(\frac{s}{3}\right) + 2u_{n-1}\left(\frac{s}{2}\right)\right)\mathrm{d}s, \quad n \ge 1. \tag{13}$$

Numerical results obtained by these approximations are reported in Table 4 and Figure 4.

We refer the interested reader for some other analytical approaches to [35 – 37] for the variational iteration technique and to [38 – 40] for the homotopy perturbation scheme and to [41] for homotopy analysis method. Also more applications of the Adomian decomposition procedure can be found in [42 – 44].

4. Conclusion

In this paper the main objective was to solve the pantograph equation of order m using the Adomian decomposition method. This method is relatively straightforward to apply at least with the assistance of a symbolic computation package (such as the well-known software Maple) and in many cases produces a series that converges rapidly to the known exact solution. The numerical results obtained in this work show a very good performance of the present procedure.

Acknowledgements

The authors are very grateful to four reviewers for carefully reading the paper and for their comments.

[1] J. R. Ockendon and A. B. Taylor, Proc. Roy. Soc. London, Ser. A **322**, 447 (1971).
[2] C. T. H. Baker and E. Bukwwar, Elec. Transac. Numer. Anal. **11**, 131 (2000).
[3] Z. Fan, M. Z. Liu, and W. Cao, J. Math. Anal. Appl. **325**, 1142 (2007).
[4] N. Guglielmi and M. Zennaro, IMA J. Numer. Anal. **23**, 421 (2003).
[5] E. Ishiwata and Y. Muroya, Appl. Math. Comput. **187**, 741 (2007).
[6] T. Koto, Numer. Math. **84**, 233 (1999).
[7] D. S. Li and M. Z. Liu, J. Harbin-Inst. Tech. **32**, 1 (2000).
[8] M. Z. Liu and D. S. Li, Appl. Math. Comput. **155**, 853 (2004).
[9] M. Z. Liu, Z. W. Yang, and G. D. Hu, BIT **45**, 743 (2005).
[10] Y. Liu, Appl. Numer. Math. **24**, 309 (1997).
[11] A. Tamasan, Pure Math. Appl. **9**, 215 (1998).
[12] C. J. Zhang and G. Sun, J. Comput. Math. **22**, 447 (2004).
[13] J. J. Zhao, Y. Xu, H. X. Wang, and M. Z. Liu, Appl. Math. Comput. **181**, 1170 (2006).
[14] G. Adomian, J. Math. Anal. Appl. **135**, 501 (1988).
[15] G. Adomian and R. Rach, Nonlinear Analysis TMA **23**, 615 (1994).
[16] A. M. Wazwaz and A. Gorguis, Appl. Math. Comput. **154**, 609 (2004).
[17] A. M. Wazwaz, Bull. Inst. Math. Acad. Sinica **29**, 125 (2001).
[18] A. M. Wazwaz, Chaos, Solitons, and Fractals **12**, 2283 (2001).
[19] A. M. Wazwaz and S. A. Khuri, Appl. Math. Comput. **80**, 287 (1996).
[20] M. Dehghan, A. Hamidi, and M. Shakourifar, Appl. Math. Comput. **189**, 1034 (2007).
[21] T. Achouri and K. Omrani, Commun. Nonlinear Sci. Numer. Simul. **14**, 2025 (2009).
[22] E. Yee, Appl. Math. Comput. **56**, 1 (1993).
[23] M. Tatari and M. Dehghan, Phys. Scripta **72**, 345 (2005).
[24] D. Lesnic, Chaos, Solitons, and Fractals, **28**, 776 (2006).
[25] M. El-Shahed, J. Fract. Calc. **24**, 23 (2003).
[26] J. S. Kamdem and Z. Qiao, Chaos, Solitons, and Fractals, **31**, 437 (2007).
[27] K. Haldar, Appl. Math. Lett. **9**, 109 (1996).
[28] A. M. Wazwaz, Appl. Math. Comput. **166**, 638 (2005).
[29] D. Kaya and K. Al-Khaled, Phys. Lett. A **363**, 433 (2007).
[30] M. Dehghan and M. Tatari, Math. Prob. Eng. **2006**, 1 (2006).
[31] S. Momani and K. Al-Khaled, Appl. Math. Comput. **162**, 1351 (2005).
[32] M. Tatari and M. Dehghan, Math. Comput. Modelling, **45**, 639 (2007).
[33] M. Dehghan Int. J. Comput. Math. **81**, 979 (2004).
[34] N. Himoun, K. Abbaoui, and Y. Cherruault, Kybernetes **10**, 423 (1999).
[35] A. Saadatmandi and M. Dehghan, Z. Naturforsch. **64a**, 783 (2009).
[36] M. Tatari and M. Dehghan, Comput. Math. Appl. **58**, 2160 (2009).
[37] M. Dehghan and A. Saadatmandi, Chaos, Solitons, and Fractals **41**, 1448 (2009).
[38] F. Shakeri and M. Dehghan, Math. Comput. Modelling **48**, 685 (2008).
[39] A. Saadatmandi, M. Dehghan, and A. Eftekhari, Nonlinear Analysis: Real World Appl. **10**, 1912 (2009).
[40] M. Dehghan and F. Shakeri, Prog. Electromagn. Res. PIER **78**, 361 (2008).
[41] M. Dehghan, J. Manafian and A. Saadatmandi, Num. Met. Part. Diff. Eqs. **26**, 448 (2010).
[42] M. Dehghan, M. Shakourifar, and A. Hamidi, Chaos, Solitons, and Fractals **39**, 2509 (2009).
[43] M. Dehghan and F. Shakeri, Numer. Methods Part. Diff. Eqs. **25**, 1238 (2009).
[44] M. Dehghan and R. Salehi, Numer. Methods Part. Diff. Eqs. **26**, 702 (2010).